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Reshaping Of the Pitch Curve with Convex Points Driven By Rotation Time Constraint for *N*-Lobed Noncircular Gear

Xin ZHANG[1, 2], Chang'an CHEN[2*], Ruiqin LI[3*]

In order to improve the transmission smoothness of noncircular gear, a novel pitch curve reshaping model for the pitch curve with convex points of N-lobed noncircular gear (N-LNG) is proposed. The model is based on fundamentals of the calculus of variations and considers minimal mean kinetic energy under the prescribed time constraint. Additionally, the reshaping algorithm for the pitch curve with convex points and the relation between prescribed time and reshaping accuracy are deduced and analyzed, respectively. Reshaping examples demonstrate the validity of the proposed reshaping model and algorithm.
***Keywords**: Noncircular gear, pitch curve reshaping, design defect, rotation time constraint.*

I. Introduction

Noncircular gears has been used in printing machine, slotting machine, flow meters and other mechanical products to replace the traditional slot wheel, cam, ratchet, or connecting-rod mechanism, to realize the variable speed , differential speed, or intermittent movement of mechanical products, differential or intermittent motion because of compact structure , stable transmission , high precision and easy to achieve dynamic balance [1-4]. Specially, a method of constructing *N*-lode noncircular gear pitch curves with Bézier, B-spline, minimal rotary inertia and minimal mean kinetic energy characteristics have been proposed in Ref. [5-7]. Conjugate pitch curves of external and internal meshing noncircular gears by the common plane curve with unidirectional continuous rotary characteristics (ellipse, eccentric circle, pascal curve, etc.) as the pitch curve of noncircular gear have been given by Liu, Wu and Li [8]. A general generation method of pitch curves for *N*-lobed elliptical gears resorting to the basic ellipse have been proposed by Figliolini, Lanni and Ceccarelli [9], so that the number of speed cycles per revolution can be increased.

Above pitch curves of noncircular gears were constructed by existing plane curves, it has been unable to meet design requirements of specific non-uniform speed transmission with the extensive application of noncircular gears in mechanical products [10-11]. This limitation has gradually been found by some researchers: A novel design method of pitch curve with epicycle constraint constructed by plane regular *N*-curved polygon for *N*-lobed noncircular gear have been presented by Yao [12]. The pitch curves with steepest rotation characteristic for noncircular gears have been synthesized, and this research has been applied to modify the pitch curve with discontinuity points [13-14].

However, according to the former researches, we know that the difficulty in manufacturing is increased for *N*-lobed noncircular gears that possess the pitch curves with convex (or concave) points [6-7, 12-13, 15-16]. In this paper, therefore, the calculus of variations is used to establish a pitch curve reshaping model and algorithm for the pitch curve with convex points for *N*-LNG, which considers minimal mean kinetic energy with prescribed time constraint because smaller mean kinetic energy can improve the transmission smoothness of noncircular gears. Numerical examples are shown to validate the proposed reshaping model and algorithm.

II. Reshaping model for the pitch curve with convex points based on calculus of variations

As shown in Fig. 1, the fixed coordinate system $\Gamma(o\text{-}xy)$ rigidly connected with the rotation center o of the pitch curve $r(\theta)$ with convex points c_1, c_2, ..., c_N for *n*-lobe noncircular gear, and the polar angle θ is measured counterclockwise from the positive direction *x*-axis.

[1] School of Intelligent Manufacturing and Electronic Engineering, Wenzhou University of Technology. Wenzhou, Zhejiang, 325035, China, e-mail: zhangxin891221@163.com
[*2] Yalong Intelligent Equipment Group Company Limited, Yongjia Industry Zone, Wenzhou, Zhejiang, 325105, China. e-mail: 282909794@qq.com
[*3] School of Mechanical Engineering, North University of China, Taiyuan, Shanxi, 030051, China. e-mail: 1085223859@qq.com

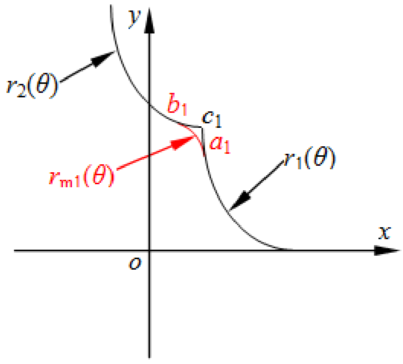

Fig. 1 Sketch of the reshaping model for pitch curve with convex points for *N*-LNG

Further, the polar equation for the pitch curve $r(\theta)$ with convex points and its conjugated external noncircular gear pitch curve $r_e(\theta_e)$ can be obtained [14]

$$r(\theta)=\begin{cases} r_1(\theta),\ \theta\in[0,\frac{2\pi}{n}] \text{ and } r_1(0)=r_1(\frac{2\pi}{n}) \\ r_2(\theta)=r_1(\theta-\frac{2\pi}{n}),\ \theta\in[\frac{2\pi}{n},\frac{4\pi}{n}] \\ \vdots \\ r_n(\theta)=r_1(\theta-\frac{2(k-1)\pi}{N}),\ \theta\in[\frac{2(k-1)\pi}{n},\frac{2k\pi}{n}] \end{cases} \tag{1}$$

$$r_e(\theta_e)=\begin{cases} r_e=A-r(\theta) \\ \theta_e=\int_0^{\theta}\frac{r(\theta)}{A-r(\theta)}d\theta \end{cases} \tag{2}$$

where n is the number of lobes of pitch curve $r(\theta)$, A is the center distance of conjugated external meshing noncircular gears.

Referring to Fig. 1, in order to ensure that the n-lobe noncircular gear pitch curve $r(\theta)$ possesses the smoothness characteristic without cusps, assuming the two tiny section pitch curves $\overset{\frown}{ac_1}$ and $\overset{\frown}{c_1b}$ can be replaced by the plane curve $r_{m1}(\theta)$ in the small neighborhood of convex cusp c_1 ($\theta_a\le\theta_{c1}\le\theta_b$), then the pitch curve $R(\theta)$ without cusps and the constraints satisfied by the plane curve $r_{m1}(\theta)$ can be expressed, respectively

$$R(\theta)=\begin{cases} \begin{cases} r_1(\theta),\ \theta\in[\theta_b-\frac{2\pi}{n},\theta_a] \\ r_{m1}(\theta),\ \theta\in[\theta_a,\theta_b] \end{cases} \\ \begin{cases} r_2(\theta)=r_1(\theta-\frac{2\pi}{n}),\ \theta\in[\theta_b,\frac{2\pi}{n}+\theta_a] \\ r_{m2}(\theta)=r_{m1}(\theta-\frac{2\pi}{n}),\ \theta\in[\frac{2\pi}{n}+\theta_a,\frac{2\pi}{n}+\theta_b] \end{cases} \\ \quad\vdots \\ \begin{cases} r_n(\theta)=r_1(\theta-\frac{2(n-1)}{n}\pi),\ \theta\in[\frac{2(n-2)\pi}{N}+\theta_b,\frac{2(n-1)\pi}{n}+\theta_a] \\ r_{mn}(\theta)=r_{m1}(\theta-\frac{2(n-1)}{n}\pi),\ \theta\in[\frac{2(n-1)\pi}{n}+\theta_a,\frac{2(n-1)\pi}{n}+\theta_b] \end{cases} \end{cases} \tag{3}$$

$$\begin{cases} r_1(\theta_a)=r_{m1}(\theta_a),\ r_1'(\theta_a)=r_{m1}'(\theta_a) \\ r_2(\theta_b)=r_{m1}(\theta_b),\ r_2'(\theta_b)=r_{m1}'(\theta_b) \end{cases} \tag{4}$$

where a and b are tangency points that the plane curve $r_{\mathrm{m1}}(\theta)$) makes with pitch curve $r_1(\theta)$) and $r_2(\theta)$), respectively.

Furthermore, the arc length S of the plane pitch curve $r_{\mathrm{m1}}(\theta)$ and the rotation time T of noncircular gear through the plane pitch curve $r_{\mathrm{m1}}(\theta)$ at the angular velocity ω can be obtained

$$S=\int_{\theta_a}^{\theta_b}\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}d\theta \tag{5}$$

$$T=\int_{\theta_a}^{\theta_b}\frac{\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}}{\omega r_{\mathrm{m1}}(\theta)}d\theta \tag{6}$$

When the modification pitch curve $r_{\mathrm{m1}}(\theta)$ meet different constraints, two different modification models can be obtained according to the variational theory [14] and related references [15-18], as shown in Table 1, and Q refers the mass of the noncircular gear.

Table 1

Three different modification models based on the variational theory

The name of modification models	Constrains
Minimal mean kinetic energy (MMKE) modification model	$E_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]=\frac{Q\omega}{2T}\int_{\theta_a}^{\theta_b}r_{\mathrm{m1}}(\theta)\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}d\theta=\frac{Q\omega}{2T}\int_{\theta_a}^{\theta_b}F_2 d\theta$
Minimal rotary inertia (MRI) modification model	$J_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]=\frac{Q}{S}\int_{\theta_a}^{\theta_b}r_{\mathrm{m1}}^{\;2}(\theta)\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}d\theta=\frac{Q}{S}\int_{\theta_a}^{\theta_b}F_3 d\theta$

Referring to Table 1, we know that the values of the minimal mean kinetic energy (MMKE) modification model $E_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$ and the minimal rotary inertia (MRI) modification model $J_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$ depend on the value of modification pitch curves $r_{\mathrm{m1}}(\theta)$. In order to ensure that the modification models $E_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$ and $J_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$ have minimum values, the solutions $r_{\mathrm{m1}}(\theta)$ must satisfy the following condition

$$\frac{\partial F_i}{\partial r_{\mathrm{m1}}(\theta)}-\frac{d}{d\theta}\left(\frac{\partial F_i}{\partial r_{\mathrm{m1}}'(\theta)}\right)=0\ ,\ \theta\in[\theta_a,\theta_b] \tag{7}$$

where i = 1, 2, and F_1, F_2 are apply to the MMKE modification model $E_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$ and the MRI modification model $J_{\mathrm{m}}[r_{\mathrm{m1}}(\theta)]$, respectively.

The integral to both sides of Eq. (7) at the same time, we can obtain

$$F_i-r_{\mathrm{m1}}'(\theta)\left(\frac{\partial F_i}{\partial r_{\mathrm{m1}}'(\theta)}\right)=p_{i0}\ ,\ \theta\in[\theta_a,\theta_b] \tag{8}$$

where p_{10} and p_{20} are integral constants.

Substituting F_1, F_2 of Table 1 into Eq. (8), we can obtain

$$r_{\mathrm{m1}}(\theta)\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}-\frac{r_{\mathrm{m1}}(\theta)r_{\mathrm{m1}}'^{\;2}(\theta)}{\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}}=p_{10} \tag{9}$$

or

$$r_{\mathrm{m1}}^{\;2}(\theta)\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}-\frac{r_{\mathrm{m1}}^{\;2}(\theta)r_{\mathrm{m1}}'^{\;2}(\theta)}{\sqrt{r_{\mathrm{m1}}^{\;2}(\theta)+r_{\mathrm{m1}}'^{\;2}(\theta)}}=p_{20} \tag{10}$$

Assuming

$$r_{m1}'(\theta) = r_{m1}(\theta)\tan u,\ u \in [u_a, u_b] \subset (-\frac{\pi}{2},\ \frac{\pi}{2}) \tag{11}$$

Substituting Eq. (11) into Eq. (9) and Eq. (10), we can obtain

$$r_{m1}(\theta) = p_{11}\sqrt{\sec u},\ p_{11} = \sqrt{p_{10}} > 0 \text{ and } p_{10} > 0 \tag{12}$$

or

$$r_{m1}(\theta) = p_{21}\sqrt[3]{\sec u},\ p_{21} = \sqrt[3]{p_{20}} > 0 \text{ and } p_{20} > 0 \tag{13}$$

where p_{11} and p_{22} are undetermined constants.
Furthermore, along with Eq. (11), the following equations can obtain

$$d\theta = \frac{dr_{m1}(\theta)}{r_{m1}'(\theta)} = \frac{dp_{11}\sqrt{\sec u}}{r_{m1}(\theta)\tan u} = \frac{du}{2} \tag{14}$$

$$u = 2\theta + p_{12} \tag{15}$$

or

$$d\theta = \frac{dr_{m1}(\theta)}{r_{m1}'(\theta)} = \frac{dp_{21}\sqrt[3]{\sec u}}{r_{m1}(\theta)\tan u} = \frac{du}{3} \tag{16}$$

$$u = 3\theta + p_{22} \tag{17}$$

where p_{12} and p_{22} are integral constants.
Substituting the Eq. (15) and Eq. (17) into the Eq. (12) and Eq. (13), the MMKE and the MRI modification pitch curves can be respectively obtained

$$r_{m1}(\theta) = p_{11}\sqrt{\sec(2\theta + p_{12})},\ \theta \in [\theta_a, \theta_b] \tag{18}$$

$$r_{m1}(\theta) = p_{21}\sqrt[3]{\sec(3\theta + p_{22})},\ \theta \in [\theta_a, \theta_b] \tag{19}$$

where p_{i1}, p_{i2} (i = 1, 2) and θ_a, θ_b are undetermined constants.
Along with Eq. (1) and Eq. (4), the undetermined constants p_{11}, p_{12} and θ_a, θ_b of the MMKE modification pitch curves $r_{m1}(\theta)$ can be solved by the following equations.

$$\begin{cases} r_1(\theta_a) = p_{11}\sqrt{\sec(2\theta_a + p_{12})} \\ r_1(\theta_b) = p_{11}\sqrt{\sec(2\theta_b + p_{12})} \\ r_1'(\theta_a) = p_{11}\sqrt{\sec(2\theta_a + p_{12})}\tan(2\theta_a + p_{12}) \\ r_1'(\theta_b) = p_{11}\sqrt{\sec(2\theta_b + p_{12})}\tan(2\theta_b + p_{12}) \end{cases} \tag{20}$$

Similarly, the undetermined constants p_{21}, p_{22} and θ_a, θ_b of the MRI modification pitch curves $r_{m1}(\theta)$ can be solved by the following equations.

$$\begin{cases} r_1(\theta_a) = p_{21}\sqrt[3]{\sec(3\theta_a + p_{22})} \\ r_1(\theta_b) = p_{21}\sqrt[3]{\sec(3\theta_b + p_{22})} \\ r_1'(\theta_a) = p_{21}\sqrt[3]{\sec(3\theta_a + p_{22})}\tan(3\theta_a + p_{22}) \\ r_1'(\theta_b) = p_{21}\sqrt[3]{\sec(3\theta_b + p_{22})}\tan(3\theta_b + p_{22}) \end{cases} \tag{20}$$

Furthermore, the pitch curve $R(\theta)$ without cusps and its conjugated external meshing noncircular gear pitch curve $R_e(\theta_e)$ can be obtained by Eq. (3) and Eq. (13), respectively.

$$R_e(\theta_e) = \begin{cases} R_e = A - R(\theta) \\ \theta_e = \int_0^\theta \frac{R(\theta)}{A - R(\theta)} d\theta \end{cases} \tag{13}$$

III. Modification examples and contrastive analysis

Example 1 (n=2): assuming the given 2-lobe noncircular gear pitch curve $r(\theta)$ with convex points and its conjugated external noncircular gear pitch curve $r_e(\theta_e)$ are depicted in Fig. 2(a)-(b), and its corresponding pitch curve equations $r(\theta)$ and $r_e(\theta_e)$ can be expressed as, respectively

$$r(\theta)=\begin{cases} r_1(\theta) = 4 - 2\sin\theta,\ \theta \in [0,\ \pi] \\ r_2(\theta) = r_1(\theta - \pi),\ \theta \in [0,\ 2\pi] \end{cases} \tag{16}$$

and

$$r(\theta)=\begin{cases} r_1(\theta) = 4 - \sqrt{3}\sin\theta - \cos\theta,\ \theta \in [0,\ 2\pi/3] \\ r_2(\theta) = r_1(\theta - 2\pi/3),\ \theta \in [2\pi/3,\ 4\pi/3] \\ r_3(\theta) = r_1(\theta - 4\pi/3),\ \theta \in [4\pi/3,\ 2\pi] \end{cases} \tag{17}$$

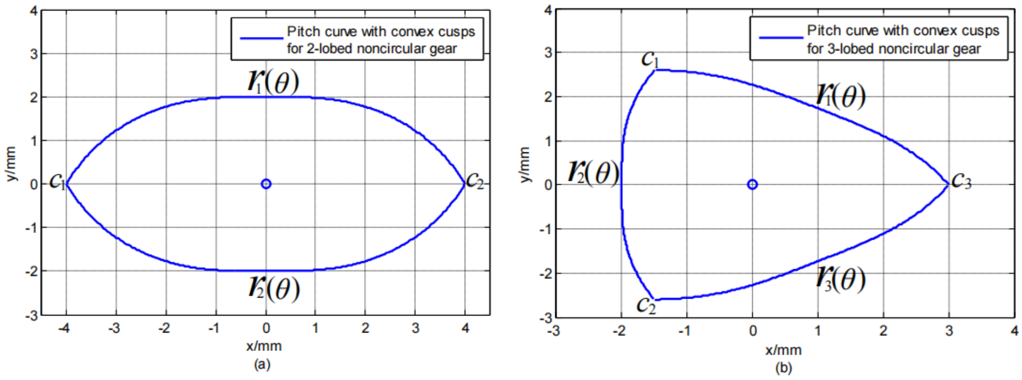

Fig. 2 Given noncircular gear pitch curve with convex points

According to the proposed reshaping algorithm, the reshaped pitch curves for the pitch curves with convex points in Fig. 3(a)-(b) are shown in Fig. 4(a)-(b), and the polar equations of reshaping pitch curves can be obtained, respectively

$$R(\theta) = \begin{cases} \begin{cases} r_1(\theta) = 4 - 2\sin\theta,\ \theta \in [0.0768,\ \pi - 0.0768] \\ r_{m1}(\theta) = \begin{cases} r_{m1}(\mu) = \sqrt{-4.6866\sec\mu + 20.0755} \\ \theta(\mu) = -0.12\ln\left|\dfrac{\tan\frac{\mu}{2} + 0.7883}{\tan\frac{\mu}{2} - 0.7883}\right| + \pi \\ s.t.\ \mu \in [0.4783,\ -0.4783] \text{ and } \theta \in [\pi - 0.0768,\ \pi + 0.0768] \end{cases} \end{cases} \\ \begin{cases} r_2(\theta) = r_1(\theta - \pi),\ \theta \in [\pi + 0.0768,\ 2\pi - 0.0768] \\ r_{m2}(\theta) = r_{m1}(\theta - \pi),\ \theta \in [2\pi - 0.0768,\ 2\pi + 0.0768] \end{cases} \end{cases} \tag{31}$$

and

$$R(\theta)=\begin{cases}\begin{cases}r_1(\theta)=4-\sqrt{3}\sin\theta-\cos\theta,\ \theta\in[0.0761,\ 2\pi/3-0.0761]\\ r_{m1}(\theta)=\begin{cases}r_{m1}(\mu)=\sqrt{-2.3597\sec\mu+10.9659}\\ \theta(\mu)=-0.1102\ln\left|\dfrac{\tan\frac{\mu}{2}+0.8036}{\tan\frac{\mu}{2}-0.8036}\right|+\dfrac{2\pi}{3}\\ s.t.\ \mu\in[0.5219,\ -0.5219]\text{ and }\theta\in[2\pi/3-0.0761,\ 2\pi/3+0.0761]\end{cases}\end{cases}\\ \begin{cases}r_2(\theta)=r_1(\theta-2\pi/3),\ \theta\in[2\pi/3+0.0761,\ 4\pi/3-0.0761]\\ r_{m2}(\theta)=r_{m1}(\theta-2\pi/3),\ \theta\in[4\pi/3-0.0761,\ 4\pi/3+0.0761]\end{cases}\\ \begin{cases}r_3(\theta)=r_1(\theta-4\pi/3),\ \theta\in[4\pi/3+0.0761,\ 2\pi-0.0761]\\ r_{m3}(\theta)=r_{m1}(\theta-4\pi/3),\ \theta\in[2\pi-0.0761,\ 2\pi+0.0761]\end{cases}\end{cases} \tag{32}$$

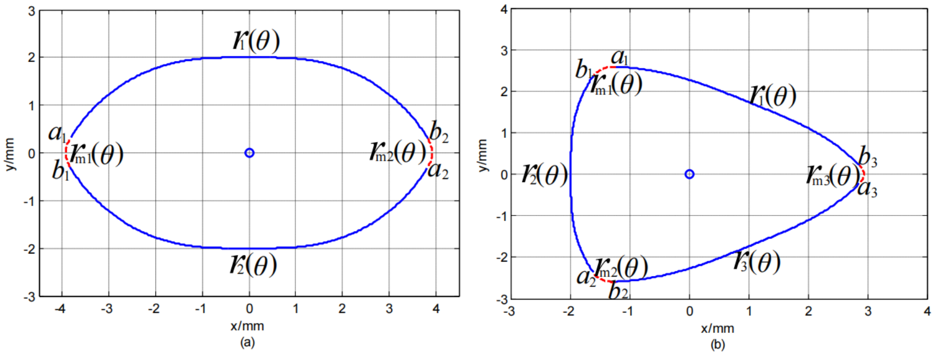

Fig. 4 Reshaped pitch curves for the pitch curves with convex points in Fig. 3

Fig. 5 is the relation the prescribed time *T* between and $\Delta\theta$, and it shows that the reshaping accuracy is increasing with gradual decrease of the prescribed time *T*. Therefore, the requirement of engineering accuracy can be satisfied by choosing appropriate time *T* for designer.

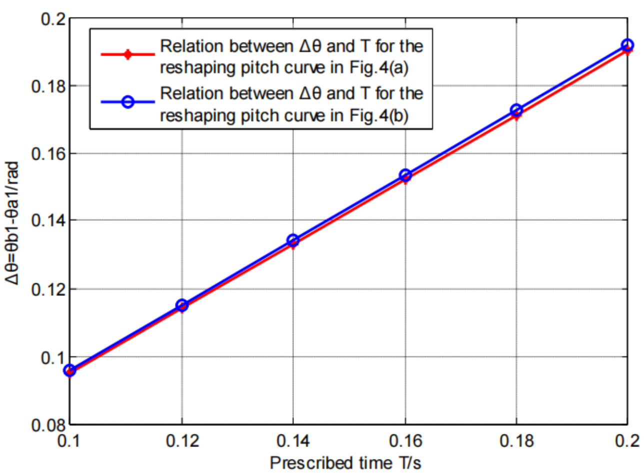

Fig. 5 Relation between the prescribed time *T* and $\Delta\theta$

IV. Conclusions

The pitch curve reshaping model and algorithm for the pitch curve with convex points of *N*-LNG are proposed, and the model can ensure that the reshaped noncircular gear possesses better smoothness. All the design examples are plotted for illustration and it shown that the reshaping model can satisfy the pitch curve design of *N*-LNG without convex points defect. Meanwhile, the reshaping accuracy can be improved by choosing appropriate time constraint.

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