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On Finding the Integer Solutions of Ternary Quadratic Diophantine Equation
$3(x^2+y^2)-5xy=36z^2$

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***ABSTRACT**: The cone represented by the ternary quadratic Diophantine equation* $3(x^2+y^2)-5xy=36z^2$ *is analyzed for its patterns of non-zero distinct integral solutions.*
KEYWORDS: *Ternary quadratic, cone, integral solutions*

I. INTRODUCTION

The Diophantine equation offers an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-13] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation $3(x^2+y^2)-5xy=36z^2$ representing homogeneous quadratic with three unknowns for determining its infinitely many non-zero integral points.

II. METHOD OF ANALYSIS

The given ternary quadratic Diophantine equation is

$$3(x^2+y^2)-5xy=36z^2 \tag{1}$$

To start with, it is seen that (1) is satisfied the following triples $(x,y,z)=$ (2444,1506, 419), (626,354,111), (852,438,157), (650,300,125), (408,192,83), (300,50,75), (244,6,69), (222, -12,67). However, we have other choices of solutions that are illustrated below:

Let us consider the linear transformations

$$x=u+v, \quad y=u-v \quad \text{where } u\neq v\neq 0 \tag{2}$$

Using (2) in (1), it gives

$$u^2+11v^2=36z^2 \tag{3}$$

Let us see the different patterns of solving the above equation (3) and thus, obtain the different choices of x, y and z satisfying (1).

Choice I:

Let us assume

$$z=z(a,b)=a^2+11b^2 \tag{4}$$

We can write 36 as

$$36=(5+i\sqrt{11})(5-i\sqrt{11}) \tag{5}$$

Using (4) and (5) in (3) and employing the method of factorization, we obtain

$$(u+i\sqrt{11}v)(u-i\sqrt{11}v)=(5+i\sqrt{11})(5-i\sqrt{11})(a+i\sqrt{11}b)^2(a-i\sqrt{11}b)^2$$

Equating the positive factors, we have

$$u+i\sqrt{11}v=(5+i\sqrt{11})(a+i\sqrt{11}b)^2$$

Comparing the real and imaginary parts, one has

$$u=5a^2-22ab-55b^2$$

$v = a^2 + 10ab - 11b^2$

Substituting the values of u and v in (2), we get

$$\left.\begin{aligned} x(a,b) &= 6a^2 - 12ab - 66b^2 \\ y(a,b) &= 4a^2 - 32ab - 44b^2 \end{aligned}\right\} \quad (6)$$

Thus, (4) and (6) represent the non-zero distinct solutions of (1).

Note 1:

The integer 36 on the R.H.S. of (3) may be written as the product of complex conjugates as shown below:

(i) $36 = \frac{(3 + i9\sqrt{11})(3 - i9\sqrt{11})}{25}$,

(ii) $36 = \frac{(19 + i7\sqrt{11})(19 - i7\sqrt{11})}{25}$

Following the above procedure, two more distinct integer solutions to (1)are obtained.

Choice II:

Write (3) as

$$u^2 + 11v^2 = 36z^2 * 1 \quad (7)$$

Let us consider 1 on the R.H.S. of (7) as

$$1 = \frac{(5 + i\sqrt{11})(5 - i\sqrt{11})}{6^2} \quad (8)$$

Substitute (4), (5) and (8) in (7). Employing the method of factorization, consider

$$u + i\sqrt{11}v = \frac{1}{6}\left(5 + i\sqrt{11}\right)\left(5 + i\sqrt{11}\right)\left(a^2 + 2i\sqrt{11}ab - 11b^2\right)$$

Equating the real and imaginary parts, we have

$$u = \frac{1}{6}(14a^2 - 220ab - 154b^2)$$

$$v = \frac{1}{6}(10a^2 + 28ab - 110b^2)$$

Substituting the values of u and v in (2), the values of x and y are

$$x(a,b) = \frac{1}{6}(24a^2 - 192ab - 264b^2)$$

$$y(a,b) = \frac{1}{6}(4a^2 - 248ab - 44b^2)$$

Replace ‘a’ by 3A and ‘b’ by 3B in x (a, b), y (a, b) and z (a, b). Then, we have

$$\left.\begin{aligned} x(A,B) &= 36a^2 - 288AB - 396B^2 \\ y(A,B) &= 6a^2 - 372AB - 66B^2 \\ z(A,B) &= 9A^2 + 99B^2 \end{aligned}\right\} \quad (9)$$

Thus (9) represents the integer solutions of (1)

Note 2:

The integer 1 on the R.H.S. of (7) may be written as the product of complex conjugates as shown below:

(i) $1=\frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100}$,

(ii) $1=\frac{(11r^2-s^2)+i\sqrt{11}2rs)(11r^2-s^2)-i\sqrt{11}2rs)}{(11r^2+s^2)^2}$,

(iii) $1=\frac{(7+i5\sqrt{11})(7-i5\sqrt{11})}{324}$,

(iv) $1=\frac{(19+i7\sqrt{11})(19-i7\sqrt{11})}{900}$,

(v) $1=\frac{(3+i9\sqrt{11})(3-i9\sqrt{11})}{900}$

Following the above procedure, five more distinct integer solutions to (1) are obtained.

Choice III:

Rewrite (3) as

$$u^2=36z^2-11v^2 \tag{10}$$

which is expressed in the form of ratio as

$$\frac{u+5z}{z+v}=\frac{11(z-v)}{u-5z}=\frac{a}{b}, b\neq 0 \tag{11}$$

Solving the above system of double equations by the method of cross-multiplication, we obtain

$$u=5a^2+22ab-55b^2$$
$$v=-a^2+10ab+11b^2$$
$$z=z(a,b)=a^2+11b^2 \tag{12}$$

Applying the values of u and v in (2), it gives

$$\left.\begin{aligned} x(a,b)&=4a^2+32ab-44b^2\\ y(a,b)&=6a^2+12ab-66b^2\end{aligned}\right\} \tag{13}$$

Thus (12) and (13) represent the integer solution of (1).

Note 3:

Observe that (10) may also be expressed in the ratio form as below:

(i) $\frac{u+5z}{z+v}=\frac{11(z-v)}{u-5z}=\frac{a}{b}, b\neq 0$

(ii) $\frac{u+5z}{z-v}=\frac{11(z+v)}{u-5z}=\frac{a}{b}, b\neq 0$

(iii) $\frac{u+5z}{11(z+v)}=\frac{(z-v)}{u-5z}=\frac{a}{b}, b\neq 0$

(iv) $\frac{u+5z}{11(z-v)}=\frac{(z+v)}{u-5z}=\frac{a}{b}, b\neq 0$

Following the above procedure, four more distinct integer solutions to (1) are obtained.

Choice IV:

Rewrite (3) as

$$u^2=36z^2-11v^2 \tag{14}$$

Consider

$$z=M+11N, v=M+36N, u=5U \tag{15}$$

Now (14) implies

$$U^2 = M^2 - 396N^2 \qquad (16)$$

satisfied by

$$M = r^2 + 396s^2, U = r^2 - 396s^2, N = 2rs$$

In view of (15), we have

$$u = 5r^2 - 1980s^2, v = r^2 + 396s^2 + 72rs \qquad (17)$$

and $z = r^2 + 396s^2 + 22rs$ (18)

Using (17) in (2), one has

$$x = 6r^2 - 1584s^2 + 72rs$$

$$y = 4r^2 - 2376s^2 - 72rs$$

along with (18), represent the integer solution of (1).

Note 4:

Note that (16) may be represented as the system of double equations as shown in Table 1 below:

Table 1: System of double equations

SYSTEM	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**IX**
M+U	$198N^2$	$99N^2$	$66N^2$	$33N^2$	$11N^2$	$9N^2$	$3N^2$	$2N^2$	N^2
M-U	2	4	6	12	36	44	132	198	396

Solving each of the above system of double equations, the values of M, N & U are obtained. From (15) and (2) the integer solutions of (1) are found. For brevity, the integer solutions thus obtained are exhibited below:

Solutions from system I:

$$x = 594N^2 + 36N - 4, y = 396N^2 - 36N - 6, z = 99N^2 + 11N + 1$$

Solutions from system II:

$$x = 1188k^2 + 72k - 8, y = 792k^2 - 72k - 12, z = 198k^2 + 22k + 2$$

Solutions from system III:

$$x = 198N^2 + 36N - 12, y = 132N^2 - 36N - 18, z = 33N^2 + 11N + 3$$

Solutions from system IV:

$$x = 396k^2 + 72k - 24, y = 264k^2 - 72k - 36, z = 66k^2 + 22k + 6$$

Solutions from system V:

$$x = 132k^2 + 72k - 72, y = 88k^2 - 72k - 108, z = 22k^2 + 22k + 18$$

Solutions from system VI:

$$x = 108k^2 + 72k - 88, y = 72k^2 - 72k - 132, z = 18k^2 + 22k + 22$$

Solutions from system VII:

$$x = 36k^2 + 72k - 264, y = 4k^2 - 72k - 396, z = 6k^2 + 22k + 66$$

Solutions from system VIII:

$$x = 6N^2 + 36N - 396, y = 4N^2 - 36N - 594, z = N^2 + 11N + 99$$

Solutions from system IX:

$x = 12k^2 + 72k - 792, y = 8k^2 - 72k - 1188, z = 2k^2 + 22k + 198$

III. CONCLUSION

In this paper, we have made an attempt to obtain all integer solutions to (1). As (1) is symmetric in x, y, z, it is to be noted that, if (x, y, z) is any positive integer solution to (1), then the triples $(x, y, -z)$, $(-x, -y, z)$, $(-x, -y, -z)$ also satisfy (1). To conclude, one may search for integer solutions to other choices of homogeneous cones.

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