International Journal of Precious Engineering Research and Applications (IJPERA)
www.ijpera.com ISSN: 2456-2734,
Volume 4, Issue 3 (May-June) 2019), PP. 01-04

Two constraint preconditioners for generalized saddle point problems+

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ABSTRACT. *In this paper, we consider two new constraint preconditioners for generalized saddle point problems. The eigenvalue distribution of the related preconditioned matrix is discussed in detail. Theoretical analysis shows that all the eigenvalues of the preconditioned matrix are strongly clustered.*
KEYWORDS: *Generalized saddle point problems; eigenvalue; constraint preconditioner*

Date of Submission: 11-06-2019 Date of acceptance: 28-06-2019

I. INTRODUCTION

We consider the generalized saddle point problems

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \qquad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $B, C \in \mathbb{R}^{m\times n}$, $m \le n$. In this paper, we always assume that $\mathbf{A}$ in (1) is nonsingular and the matrix A is singular with high nullity, i.e., the solution of the generalized saddle point problems (1) exists and is unique. That is, the generalized saddle point problems (1) is important and appears in many different applications of scientific computing, one can see [1] for a comprehensive survey.

Recently, a great deal of effort has been invested in solving the generalized saddle point problems. Most of the work has been aimed at developing some effective preconditioning techniques for generalized saddle point problems. In general, there at least exist five classes of preconditioners to improve the convergence rate of Krylov subspace methods for solving generalized saddle point problems： diagonal preconditioner, triangular preconditioner, constraint preconditoner, HSS preconditioner, shift-preconditioner, see [1-6].

In this paper, two new constraint preconditioners for the generalized saddle point problems are presented and the eigenvalue distribution of the preconditioned matrices is given. If the nullity of the (1,1) block in $\mathbf{A}$ of the generalized saddle point problems (1) takes its highest possible value, then some precisely distinct eigenvalues of the preconditioned matrix can be obtained.

1. Preconditioners and Spectrum Analysis

To conveniently discuss the block triangular preconditioners for solving (1), without further illustration, we always assume that $\mathbf{A}$ is nonsingular. From [2,3], the following two lemmas are required.

Lemma 2.1 The nonsymmetric coefficient matrix

$$\mathbf{A} = \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}$$

is nonsingular if and only if the following conditions are satisfied:

$$rank(B) = rank(C) = m, \mathrm{N}(A) \cap \mathrm{N}(C) = \{0\} \ and\ \mathrm{N}(A^T) \cap \mathrm{N}(B) = \{0\},$$

where $\mathrm{N}(\cdot)$ denotes the null space of a matrix.

Lemma 2.2 The nonsymmetric coefficient matrix

$$\mathbf{A} = \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}$$

is nonsingular, then the rank of the matrix A is at least $n-m$, and hence its nullity is at most m.

Next, the following two augmentation block constraint preconditioners are considered

+ This research of this author was supported by NSFC (11301009).
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$$P_- = \begin{pmatrix} A + B^T W^{-1} C & 3B^T \\ C & -W \end{pmatrix} \text{ and } P_+ = \begin{pmatrix} A + 4B^T W^{-1} C & B^T \\ C & W \end{pmatrix}$$

where $W \in \mathbb{R}^{m \times m}$ is nonsingular and such that $A + B^T W^{-1} C$ is nonsingular.

The following theorem provides the spectrum results of the preconditioned matrix $P_-^{-1}\mathbf{A}$.

Theorem 2.1 Assume that $\mathbf{A}$ is nonsingular and the matrix A is singular with nullity s. Then $\lambda = 1$ is an eigenvalue of $P_-^{-1}\mathbf{A}$ of geometric multiplicity $n-m$, and $\lambda = 1/2$ is an eigenvalue of geometric multiplicity s. The remaining $2m-s$ eigenvalues satisfy

$$\lambda = \frac{\sqrt{4\mu + 1} \pm 1}{2},$$

where μ are some $2m-s$ generalized eigenvalues of the generalized eigenvalues problem

$$B^T W^{-1} C x = \mu A x. \tag{2}$$

Let $\{z_i\}_{i=1}^{n-m}$ be a basis of $\mathrm{N}(C)$ and $\{x_i\}_{i=1}^{s}$ a basis of $\mathrm{N}(A)$. Then the vectors $\left[z_i^T, 0^T\right]^T$, $i = 1, \cdots, n-m,$ are linearly independent eigenvectors associated with $\lambda = 1$, and the vectors $\left[x_i^T, -(W^{-1} C x_i)^T\right]^T$ $(i = 1, \cdots, s)$ are linearly independent eigenvectors associated with $\lambda = 1/2$.

Proof. Let λ be an eigenvalue of $P_-^{-1}\mathbf{A}$ and $\left[u^T, v^T\right]^T$ be the corresponding eigenvector. Then

$$\begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} A + B^T W^{-1} C & 3B^T \\ C & -W \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

or equivalently,

$$Au + B^T v = \lambda (A + B^T W^{-1} C) u + 3\lambda B^T v \tag{3}$$

$$Cu = \lambda C u - \lambda W v \tag{4}$$

As $\mathbf{A}$ is nonsingular, $\lambda \neq 0$. From (4), we can get that

$$v = \frac{\lambda - 1}{\lambda} W^{-1} C u. \tag{5}$$

Substituting (5) into (3), we get

$$Au + \frac{\lambda - 1}{\lambda} B^T W^{-1} C u = \lambda (A + B^T W^{-1} C) u + 3\lambda \cdot \frac{\lambda - 1}{\lambda} B^T W^{-1} C u.$$

By simple computations, we obtain that

$$(1-\lambda)\lambda A u = (4\lambda^2 - 4\lambda + 1) B^T W^{-1} C u. \tag{6}$$

If $u \in \mathrm{N}(C)$, then (6) implies that

$$(1-\lambda)\lambda A u = 0.$$

Further, we can get that $\lambda = 1$ and $\left[u^T, 0^T\right]^T$ is its eigenvector. Thus, if $\{z_i\}_{i=1}^{n-m}$ is a basis of $\mathrm{N}(C)$, then the vectors $\left[z_i^T, 0^T\right]^T$, $i = 1, \cdots, n-m,$ are linearly independent eigenvetors associated with the eigenvalue $\lambda = 1$.

If $u \in \mathrm{N}(A)$, then (6) implies that

$$(4\lambda^2 - 4\lambda + 1) B^T W^{-1} C u = 0,$$

from which we obtain that $\lambda = 1/2$ and $\left[u^T, -(W^{-1} C u)^T\right]^T$ are the eigenvetors. Thus, let $\{x_i\}_{i=1}^{s}$ be a basis of $\mathrm{N}(A)$, then the vectors $\left[x_i^T, -(W^{-1} C x_i)^T\right]^T$, $i = 1, \cdots, s,$ are linearly independent eigenvetors associated with the eigenvalue $\lambda = 1/2$.

If $u \notin \mathrm{N}(A)$ and $u \notin \mathrm{N}(C)$, based on (2) and (6), we get

$$(1-\lambda)\lambda=(4\lambda^2-4\lambda+1)\mu$$

or

$$(4\mu+1)\lambda^2-(4\mu+1)\lambda+\mu=0.$$

Therefore,

$$\lambda=\frac{\sqrt{4\mu+1}\pm 1}{2},$$

which completes the proof. □

Theorem 2.1 shows that the higher the nullity of A is, the stronger the eigenvalues of $P_-^{-1}\mathbf{A}$ are clustered. When the nullity of A is m, its at most value from Lemma 2.2, we have the following result.

Corollary 2.1 Assume that $\mathbf{A}$ is nonsingular and that its (1,1) block A is singular with nullity m. Then $\lambda=1$ is an eigenvalue of $P_-^{-1}\mathbf{A}$ of geometric multiplicity $n-m$, and $\lambda=1/2$ is an eigenvalue of geometric multiplicity m.

Based on the results in Corollary 2.1, we know that a preconditioned minimal residual Krylov iterative method such as GMRES with the preconditioner P_- converges with two iterations.

Similarly, we can obtain the following spectrum results of the preconditioned matrix $P_+^{-1}\mathbf{A}$

Theorem 2.2 Assume that $\mathbf{A}$ is nonsingular and the matrix A is singular with nullity s. Then $\lambda=1$ is an eigenvalue of $P_+^{-1}\mathbf{A}$ of geometric multiplicity $n-m$, and $\lambda=-1$ and $\lambda=1/3$ are two eigenvalues of geometric multiplicity s. The remaining $2m-2s$ eigenvalues satisfy

$$\lambda=\frac{1-2\mu\pm\sqrt{1+16\mu^2}}{2(3\mu+1)}$$

where μ are some $2m-2s$ generalized eigenvalues of the generalized eigenvalues problem

$$B^T W^{-1} C x=\mu A x.$$

Corollary 2.2 Assume that $\mathbf{A}$ is nonsingular and the matrix block A is singular with nullity m. Then $\lambda=1$ is an eigenvalue of $P_+^{-1}\mathbf{A}$ of geometric multiplicity $n-m$, and $\lambda=-1$ and $\lambda=1/3$ are two eigenvalues of geometric multiplicity m.

Based on the results in Corollary 2.2, we know that a preconditioned minimal residual Krylov iterative method such as GMRES with the preconditioner P_+ converges with three iterations.

II. CONCLUSIONS

In this paper, two new constraint preconditioners for generalized saddle point problem are presented and the spectrum distribution of corresponding preconditioned matrices are discussed. Theoretical analysis shows that the eigenvalues of the preconditioned matrix $P_-^{-1}\mathbf{A}$ are 1 and $1/2$ when the matrix block A in $\mathbf{A}$ is singular with nullity m, the eigenvalues of the preconditioned matrix $P_+^{-1}\mathbf{A}$ are 1, -1 and $1/3$ when the matrix block A in $\mathbf{A}$ is singular with nullity m.

REFERENCES

[1]. M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, Acta Numerica, 14 (2005) 1-137.

[2]. Z.H. Cao, Augmentation block precondtioners for saddle point-type matrices with singular (1,1) blocks, Numerical Linear Algebra with Its Applications, 15 (2008) 515-533.

[3]. C.-X. Li, S.-L. Wu, Two block triangular preconditioners for asymmetric saddle point problems, Applied Mathematics and Computation, 269 (2015) 456-463

[4]. C.-X. Li, S.-L. Wu, A.-Q. Huang, A relaxed splitting preconditioner for saddle point problems，Journal of Numerical Mathematics, 23(4) (2015) 361-368

[5]. C.-X. Li, S.-L. Wu, Eigenvalue distribution of relaxed mixed constraint preconditioner for saddle point problems, Hacettepe Journal of Mathematics and Statistics, 45 (2016) 1705-1718

[6]. C.-X. Li, S.-L. Wu, Some new estimates on the complex eigenvalues of the HSS preconditioned matrix，Applied Mathematics and Computation, 248 (2014) 519-524

[7]. Y. Cao, J. Dua, Q. Niu, Shift-splitting preconditioners for saddle point problems, Journal of Computational and Applied Mathematics, 272 (2014) 239-250